Annals of Fuzzy Mathematics and Informatics
Volume 1, No. 1, (January 2011), pp. 13- 23
ISSN 2093-9310
http://www.afmi.or.kr

@FMI

Some remarks on strong semilattices of monoids

S. K. Elagan

Received 10 September 2010; Revised 29 September 2010; Accepted 25 October 2010

Abstract. In this paper we discuss the structure of certain kinds of strong semilatticess of monoids, the so called free semilattices of monoids to obtain its properties analogous to the basic known properties of free monoids.

2010 AMS Classification: 20M12, 15A66

Keywords: Partial monoid, Free partial monoid, Semilattices of monoids.

1. Introduction

Perhaps one may recognize that the theory of languages and automata is based on certain properties of particular subsets of free monoids the so called rational sets. Besides the uniqueness of factorization of words in free monoids plays a central role in developing such a theory. It is thus not surprising that characterizations of submonoids of free monoids which are also free has obtained considerable interest from both semigroups and languages theorists. This naturally lead to study and characterizing certain types of codes, i.e. the bases or generating sets of free monoids. Prefix codes in particular has its significant role in the study of rational languages and finite automata . In nature, there are different languages expressed in terms of different (and disjoint) alphabets. Interaction between different languages may be viewed as mappings (or translators) between the corresponding alphabets. Formally, a natural model of the situation may have the structure of certain kinds of strong semilattices of monoids, the so called free semilattices of monoids. In the present work we discuss this structure to obtain its properties analogous to the basic known properties of free monoids..

2. Preliminaries

For sake of reference and fixing notation we cite here some basic definitions and results needed for our work.

A semilattice is a pair $(S, \leq)$ where S is a set and $\leq$ is a partial ordering on S (i.e. $\leq$ is a reflexive, antisymmytric and transitive relation on S), such that every pair of elements $a, b \in S$ has a greatest lower bound $a \wedge b$ in S.

A semigroup S is called a band if every element $a \in S$ is idempotent i.e. $aa = a$ (or $a^2 = a$). Note that a semigroup S is a commutative band iff S (with the same partial ordering) is a semilattice. Actually if S is a commutative band, then the relation $\leq$ define on S by $a \leq b$ iff $ab = a$ turns S into a semilattice where for every pair $a, b \in S$, $a \wedge b$ is given by ab (the product of a, b in S). Conversely, if $(S, \leq)$ is a semilattice, then S is a commutative band with the operation in S defined by $\forall$ $a, b \in S$, $ab = a \wedge b$ $(= b \wedge a) = ba$. (Thus in particular $a^2 = a = a \wedge a = a$).

Let Y be a semilattice and let $\{S_\alpha : \alpha \in Y\}$be a collection of (disjoint) semigroups of particular type $\mathcal{F}$.

Then the disjoint union $S = \underset{\alpha \in Y}{\cup} S_\alpha$ is called a strong semilattice of semigroups S_α , $\alpha \in Y$ if for all α, $\beta \in Y$ with $\alpha \geq \beta$ there exists a homomorphism

$$\varphi_{\alpha,\beta} : S_\alpha \to S_\beta$$

such that $\varphi_{\alpha,\alpha}$ is the identical homomorphism, and for $\alpha \geq \beta \geq \gamma$ in Y,

$$\varphi_{\beta,\gamma} \circ \varphi_{\alpha,\beta} = \varphi_{\alpha,\gamma}.$$

We may write $S = [Y, S_\alpha, \varphi_{\alpha,\beta}\ ]$ to indicate that S is a strong semilattice Y of semigroups S_α, $\alpha \in Y$.

If $S = [Y, S_\alpha, \varphi_{\alpha,\beta}\ ]$, then there is a (unique) operation on S that extends the binary operation of S_α for every $\alpha \in Y$, given by, for $a, b \in S$, say $a \in S_\alpha$, $b \in S_\beta$

$$ab = \varphi_{\alpha,\alpha\beta}\ a \circ \varphi_{\beta,\alpha\beta}\ b.$$

Here the operation in RHS is the multiplication in $S_{\alpha\beta}$ of the elements $\varphi_{\alpha,\alpha\beta}a$, $\varphi_{\beta,\alpha\beta}b$ in $S_{\alpha\beta}$.

In particular a strong semilattice of groups is a Clifford semigroup (i.e. a regular semigroup with centeral idempotents). For our work we discuss the structure of strong semilattices of monoids. In the rest of this section we cite from [5] some requried materials.

A semigroup S is called an $\varepsilon-$semigroup (in words, epsilon semigroup) if it is equipped with a unary operation $\varepsilon : S \to S$, $x \mapsto \varepsilon_x$, with the following axioms: for all $x, y \in S$

(PM1) ε_x is idempotent (i.e. $\varepsilon_x\ \varepsilon_x = \varepsilon_x$)

(PM2) $\varepsilon_{(\varepsilon_x)} = \varepsilon_x$ (i.e. the operation ε is idempotent)

(PM3) $\varepsilon_x\ x = x\ \varepsilon_x = x$

(PM4) $\varepsilon_{(xy)} = \varepsilon_x\ \varepsilon_y$

The element ε_x is called the partial identity of x. The subset of idempotents in an $\varepsilon-$semigroup S is denoted by $E(S)$, and the set of all partial identities in S, $\{\varepsilon_x :\ x \in S\}$ is denoted by $\varepsilon(S)$.

If S is an $\varepsilon-$semigroup, then by (PM1), $\varepsilon(S) \subset E(S)$ and so by (PM4), $\varepsilon(S)$ is idempotent subsemigroup of S.

A subset B of an $\varepsilon-$semigroup S is an $\varepsilon-$subsemigroup of S, if B is a subsemigroup of S and $\varepsilon_b \in B$ for every $b \in B$.

A mapping φ of an $\varepsilon-$semigroup S into an $\varepsilon-$semigroup T is an $\varepsilon-$homomorphism if it preserves the operations of S, i.e. $\varphi(xy) = \varphi(x)\varphi(y)$, $x, y \in S$ and $\varepsilon_{(\varphi x)} = \varphi(\varepsilon_x)$, for all $x \in S$. Hence $\varepsilon_{(\varphi x)}$ is the partial identity in T of φx, and ε_x is the partial identity in S (of x).

Clearly, the variety of $\varepsilon-$semigroups contains monoids, bands, and Clifford semigroups. Those $\varepsilon-$semigroup S for which $\varepsilon(S)$ is in the center of S have a structure theorem of strong sort. First, a definition:

An $\varepsilon-$semigroup S is called a *partial monoid* if (PM5) ε_x is central (for all $x \in S$)

If S is a partial monoid, then an $\varepsilon-$subsemigroup of S is called a subpartial monoid of S. A partial monoid homomorphism is defined similarly.

In view of (PM2), we have $\varepsilon(S)$ is an idempotent $\varepsilon-$subsemigroup (subpartial monoid) of S whenever S is an $\varepsilon-$semigroup (partial monoid). In particular (by PM5), if S is a partial monoid, then $\varepsilon(S)$ is a commutative semigroup of idempotents, i.e. a semilattice (with the usual partial ordering $\varepsilon_x \leq \varepsilon_y$ iff $\varepsilon_x\, \varepsilon_y = \varepsilon_x$)

Theorem 2.1. *The following two statements about a semigroup S are equivalent.*

(A) *S is a partial monoid.*

(B) *S is a strong semilattice of monoids.*

Remark. *The above theorem shows that if S is a partial monoid, then S is a strong semilattice of monoids*

$$S = \left[\varepsilon(S), S_{\varepsilon_x},\ \varphi_{\varepsilon_x,\varepsilon_y}\right]$$

We have for ε_x in the semilattice $\varepsilon(S)$, S_{ε_x} is the maximal monoid $\{y \in S : \varepsilon_y = \varepsilon_x\}$ with the identity ε_x and for $\varepsilon_x \geq \varepsilon_y$ (i.e. $\varepsilon_x\, \varepsilon_y = \varepsilon_y$)

$$\varphi_{\varepsilon_x,\varepsilon_y} : S_{\varepsilon_x} \to S_{\varepsilon_y},\ a \mapsto a\varepsilon_y$$

Conversely, if S is a strong semilattice of monoids $S = [\Upsilon, S_\alpha,\ \psi_{\alpha,\beta}]$, then S is a partial monoid with $\varepsilon-$ operation for $x \in S$, say $x \in S_\alpha$, $\varepsilon_x = e_\alpha$ where e_α is the identity of the monoid S_α.

In [5] some topological and categorical aspects of partial monoids (not needn't in our present work) are obtained as well as a representation theorem says that every partial monoid S is empeddablle in a certain partial monoid of partial mappings in analogy with the same sort of theorem known for strong semilattices of groups i.e. Clifford semigroups (see [1,2,5], where they are called partial groups) and for strong semilattices of rings (see[3,4,7], where they are called partial rings).

In [5] examples are given to show that :

For an $\varepsilon-$ semigroup S, $\varepsilon(S)$ may be a proper subset of $E(S)$, i.e. an idempotent in S may not be a partial identity.

There may exist different $\varepsilon-$ semigroup structures on the same semigroup S.

There may exist an $\varepsilon-$ semigroup S which is not a partial monoid (and hence not a monoid).

Non trivial partial monoids exists, i.e. partial monoids which are not monoids (These are introduced, in particular by partial mappings (from sets to monoids). Less trivially, every partial monoid S is embeddablle in a certian partial monoid of partial mappings [5, Theorem 3.4].

As shown in [5], it is easy to observe that the class of partial monoids is a variety $(\Omega,\ E)$ of algebras for some generator domain Ω, and set of equations E, whence free partial monoids exist. In the next section, we introduce explicit construction of free partial monoids and develop the basic properties and characterization of them.

Our references are, in semigroups, in general, [12,13,14,15,18,19], in semilattices of monoids [5,6,8,9,10], and in free monoids and codes [9,11,16,17].

3. Free partial monoids. Construction and basic characterizations

Let A be a non empty (not necessarily finite) set. For every nonempty finite subset B of A, let ε_B denote the identity element of the free monoid B^* on B. In other words, ε_B stands for the empty word in B^*. There exists a natural embedding (satisfying the usual universal property)

$$\eta_B : B \to B^*$$

Actually, η_B sends each element b in the set B to the word in B^* that consists only of one alphabet b.

Let $\varepsilon\left(A\right) = \{\varepsilon_B :\ B \text{ is a non empty finite subset of } A\}$. Partially ordered $\varepsilon\left(A\right)$ by

$$\varepsilon_C \leq \varepsilon_B \text{ if and only if } B \subset C.$$

Then $\varepsilon\left(A\right)$ with $\leq$ is clearly a semilattice, whence the greatest lower bound of any $\varepsilon_B, \varepsilon_C \ \in \varepsilon\left(A\right)$ is given by

$$\varepsilon_B \wedge \varepsilon_C = \varepsilon_{B\cup C}.$$

Equivalently, $\varepsilon\left(A\right)$ is a commutative band with binary operation given by

$$\varepsilon_B\ \varepsilon_C = \varepsilon_{B\cup C} \ \text{ for all } \varepsilon_B, \varepsilon_C \ \in \varepsilon\left(A\right),$$

and we have for all $\varepsilon_B, \varepsilon_C \ \in \varepsilon\left(A\right)$

$$\varepsilon_B\ \varepsilon_C = \varepsilon_C \text{ if and only if } \varepsilon_C \leq \varepsilon_B \text{ if and only if } B \subset C.$$

For $\varepsilon_B \geq \varepsilon_C$ in $\varepsilon\left(A\right)$, (i.e. $B \subset C$), we define a mapping

$$\varphi_{\varepsilon_B\ ,\varepsilon_C}\ : B^* \to C^*$$

as follows: For any non empty word $w \in B^*$, say $w = \eta_B\ b_1 ... \eta_B\ b_n$ $(b_i \in B)$, we set

$$\varphi_{\varepsilon_B\ ,\varepsilon_C}\ w = \eta_C\ b_1 ... \eta_C\ b_n.$$

For the empty word ε_B of B^*, we set

$$\varphi_{\varepsilon_B\ ,\varepsilon_C}\ \varepsilon_B = \varepsilon_C \qquad \text{(the empty word in } C^*\text{)}.$$

We observe that $\varphi_{\varepsilon_B\ ,\varepsilon_C}$ is a well defined mapping (since $B \subset C$) and clearly a monoid homomorphism. Actually, $\varphi_{\varepsilon_B\ ,\varepsilon_C}$ is a monoid monomorphism. It is also easy to see that $\varphi_{\varepsilon_B\ ,\varepsilon_B}$ is the identity automorphism of B^* and that

$$\varphi_{\varepsilon_C\ ,\varepsilon_D}\ \cdot \varphi_{\varepsilon_B\ ,\varepsilon_C} = \varphi_{\varepsilon_B\ ,\varepsilon_D}$$

for all ε_B , ε_C, ε_D in $\varepsilon\left(A\right)$, satisfying $\varepsilon_B\ , \geq \varepsilon_C \geq \varepsilon_D$. Summing up, we have a strong semilattice of monoids

$$FPM\left(A\right) = \langle \varepsilon\left(A\right), B^*\ ,\ \varphi_{\varepsilon_B\ ,\varepsilon_C} \rangle.$$

Whence,

$$FPM\left(A\right) = \bigcup_{\varepsilon_B \in \varepsilon(A)} B^*_{\varepsilon_B}$$

is a partial monoid, with operation (extending the operation of $B^*_{\varepsilon_B}$, $\varepsilon_B \in \varepsilon\left(A\right)$ given by , for any two elements in $FPM\left(A\right)$, say

$w_B = \eta_B\ b_1 ... \eta_B\ b_n \in B^*$ and $w_C = \eta_C\ c_1 ... \eta_C\ c_m \in C^*$

we have

$$\begin{aligned} w_B \cdot w_C &= \varphi_{\varepsilon_B\ ,\varepsilon_{B\cup C}}\ w_B \cdot \varphi_{\varepsilon_C\ ,\varepsilon_{B\cup C}}\ w_C \\ &= \eta_{B\cup C}\ b_1 ... \eta_{B\cup C}\ b_n\ \eta_{B\cup C}\ c_1 ... \eta_{B\cup C}\ c_m \in (B \cup C)^* \end{aligned}$$

We define a mapping

$$\eta : A \to FPM(A) = \bigcup_{\varepsilon_B \in \varepsilon(A)} B^*_{\varepsilon_B}$$

by

$$a \mapsto \eta_{\{a\}}\ a$$

i.e. sending each element $a \in A$ to the word in $\{a\}^*$ consisting of one letter, namely a.

Now, let S be any partial monoid and let $\psi : A \to S$ be any mapping of sets . Define

$$\bar{\psi} : FPM(A) \to S$$

as follows: Let $w \in FPM(A)$, be arbitrary. Then, there is a (unique) nonempty finite subset B of A,say $B = \{b_1, b_2, ..., b_r\}$ and a unique (may be empty) subset $\{b_{i_1}, b_{i_2}, ..., b_{i_n}\}$ of B, with

$$w = w_B = \eta_B\ b_{i_1} ... \eta_B\ b_{i_n}.$$

Define

$$\bar{\psi} w = \psi\ b_{i_1} ...\ \psi b_{i_n} \prod_{i=1}^{r} \varepsilon_{\psi b_i}$$

where $\varepsilon_{\psi b_i}$ is the partial identity of the element ψb_i in the partial monoid S. It follows that for each $w_B \in B^*$ we have

$$\varepsilon_{\left(\bar{\psi}\ w_B\right)} = \varepsilon_{\psi b_1}\ \varepsilon_{\psi b_2} ... \varepsilon_{\psi b_r} = \prod_{i=1}^{r} \varepsilon_{\psi b_i}.$$

Identifying $a \in A$ with $\eta_{\{a\}}\ a$ and $\varepsilon_{\{a\}}$ with ε_a, then the identity ε_B, viewed as the empty word in B^*, may be written

$$\begin{aligned} \varepsilon_B &= \varepsilon_{\eta_{\{b_1\}} b_1}\ \varepsilon_{\eta_{\{b_2\}} b_2} ... \varepsilon_{\eta_{\{b_r\}} b_r} \\ &= \varepsilon_{b_1}\ \varepsilon_{b_2} ... \varepsilon_{b_r} = \prod_{i=1}^{r} \varepsilon_{b_i} \end{aligned}$$

By the definition of $\bar{\psi}$, $\bar{\psi}\varepsilon_B = \prod_{i=1}^{r} \varepsilon_{\psi b_i}$, and since $\varepsilon_{(w_B)} = \varepsilon_B$, it follows that

$$\varepsilon_{\bar{\psi}\ w_B} = \bar{\psi}\ \left(\varepsilon_{(w_B)}\right).$$

Clearly, $\bar{\psi}$ is a partial monoid homomorphism, with $\bar{\psi}\left(B^*_{\varepsilon_B}\right) \subset S_{\prod_{i=1}^{r} \varepsilon_{\psi b_i}}$, where $S_{\prod_{i=1}^{r} \varepsilon_{\psi b_i}}$ is the maximal monoid in S with identity $\bar{\psi}(\varepsilon_B) = \prod_{i=1}^{r} \varepsilon_{\psi b_i}$.

For every $a \in A$, we have

$$\begin{aligned}\left(\bar{\psi}\ \eta\right)(a) &= \bar{\psi}\ (\eta\ a) = \bar{\psi}\ \left(\eta_{\{a\}}\ a\right)\\ &= \psi a \cdot \varepsilon_{\psi a} = \psi a \qquad ,\end{aligned}$$

whence, $\bar{\psi}\ \eta = \psi$, that is the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta} & FPM\,(A) \\ & \psi \searrow & \downarrow\bar{\psi} \\ & & S. \end{array}$$

commutes. If $\varphi :\ FPM\,(A) \to S$ is a partial monoid homomorphism with $\varphi\eta = \psi$, we have for any $w \in FPM\,(A)$, say $w = w_B = \eta_B\ b_{i_1}...\eta_B\ b_{i_n} \in B^*$ (with $B = \{b_1, ..., b_r\}$),

$$\begin{aligned}\varphi w &= \varphi\left(\eta_{\left\{b_{i_1}\right\}}b_{i_1}...\eta_{\{b_{i_n}\}}b_{i_n}\ \cdot \varepsilon_B\right)\\ &= \varphi\eta_{\left\{b_{i_1}\right\}}b_{i_1}...\varphi\eta_{\{b_{i_n}\}}b_{i_n}\cdot\ \varphi\left(\varepsilon_B\right)\\ &= \varphi\eta b_{i_1}...\varphi\eta b_{i_n}\cdot\ \varphi\left(\varepsilon_B\right)\\ &= \psi b_{i_1}...\psi b_{i_n}\cdot\ \varphi\left(\varepsilon_B\right)\end{aligned}$$

since φ is a partial monoid homomorphism, we have $\varphi\left(B^*_{\varepsilon_B}\right) \subset S_{\varphi(\varepsilon_B)}$. Now,

$$\varepsilon_B = \varepsilon\left({}_{\eta_{\{b_1\}}}b_1\ \varepsilon_{\eta_{\{b_2\}}}b_2...\varepsilon_{\eta_{\{b_r\}}}b_r\right)$$

thus

$$\begin{aligned}\varphi\left(\varepsilon_B\right) &= \varepsilon_{\varphi\left(\eta_{\{b_1\}}b_1\ \varepsilon_{\eta_{\{b_2\}}}b_2...\varepsilon_{\eta_{\{b_r\}}}b_r\right)}\\ &= \varepsilon_{(\varphi\eta b_1...\varphi\eta b_r)}\\ &= \varepsilon_{(\psi b_1...\psi b_r)} = \varepsilon_{\psi b_1}...\varepsilon_{\psi b_r}\\ &= \prod_{i=1}^{r}\varepsilon_{\psi b_i}.\end{aligned}$$

Thus,

$$\begin{aligned}\varphi w &= \psi b_{i_1}...\psi b_{i_n}\cdot\ \varphi\left(\varepsilon_B\right)\\ &= \psi b_{i_1}...\psi b_{i_n}\cdot\prod_{i=1}^{r}\varepsilon_{\psi b_i}\\ &= \bar{\psi}w.\end{aligned}$$

Hence $\varphi = \bar{\psi}$. Therefore $\bar{\psi}$ is the unique partial monoid homomorphism such that $\bar{\psi}\ \eta = \psi$.

Theorem 3.1. *For any non empty set A, the partial monoid $FPM\,(A)$ is (up to an isomorphism) the free partial monoid on A.*

Remark. *In the free partial monoid $FPM(A)$, the effect of multiplying a "word" w_B by an $\varepsilon_C \in \varepsilon(A)$ (for some finite subsets $B,\ C \subset A$) is nothing but transforming $w_B \in B^*$ to the word $w_{B\cup C} \in (B \cup C)^*$ having the same string of alphabets as w_B. In particular if $b \in B$, then $\eta_B b \in B^*$ may be viewed as $\eta_{\{b\}} b.\varepsilon_B$. (Hence $\eta_{\{b\}} b \in \{b\}^*$) As we identify $\eta_{\ \{b\}}\ b = \eta b$ with b, we may write $\eta_B b = b\varepsilon_B$. Thus if $w_B = \eta_B\ b_{i_1}...\eta_B\ b_{i_n}$ is a word in B^*, we may write*

$$\begin{aligned} w_B &= \eta\ b_{i_1}...\eta\ b_{i_n}\ \varepsilon_B \\ &= b_{i_1}...\ b_{i_n}\ \varepsilon_B. \end{aligned}$$

It follows that each (non empty) word w in $FPM(A)$ say $w = w_B$, for some non empty finite subset B of A,has a unique representation as product of alphabets from $B \subset A$ with ε_B.

In the rest of this section we give some characterizations of free partial monoids analogous to the known characterizations [16] of free monoids. We start with a definition.

Let M be a partial monoid (i.e. strong semilattice $[\varepsilon(A),\ M_{\varepsilon_a},\ \varphi_{\varepsilon_a,\varepsilon_b}]$ of monoids). We call a subset A of M a set of partial generators of (or partially generates) M if for every $b \in M$, with $b \neq \varepsilon_b$, there is a finite set $\{a_1, ..., a_r\} \subset A$ such that

$$b = x_1 x_2 ... x_n\ \varepsilon_{\prod_{i=1}^{r} a_i}$$

with $x_i,\ i = 1, 2, ..., n$ (possibly not all distinct) are elements of $\{a_1, ..., a_r\}$.

Theorem 3.2. *Let A be a (non empty) set, M a partial monoid and let $i : A \to M$ be an injection onto a set of partial generators of M. The following two statements are equivalent:*

(a) *M is free on $i(A)$.*
(b) *For any partial monoid M' and map $\varphi : A \to M'$, there is a unique homomorphism of partial monoids $\bar{\varphi} : M \to M'$ such that $\varphi = \bar{\varphi} i$.*

Proof. $(a) \Rightarrow (b)$. Let $\eta :\ i(A) \to M$ be the natural embedding (as in Theorem 2.1). Define $\varphi' : i(A) \to M'$ by $i(a) \mapsto \varphi(a)$, where $\varphi : A \to M'$ is a given map. φ' is a well defined map, since i is injection. By Theorem 2.1, there exists a unique partial monoid homomorphism $\bar{\varphi}' : M \to M'$ such that $\varphi' = \bar{\varphi}' \circ \eta$. We have φ' $(i(a)) = \varphi(a)$, $(a \in A)$. Thus

$$\begin{aligned} \varphi(a) &= \varphi'(i(a)) = \bar{\varphi}' \circ \eta\ (i(a)) \\ &= \bar{\varphi}'\left(i(a)\ \varepsilon_{\eta\ i(a)}\right) \\ &= \bar{\varphi}'(i(a))\ \varepsilon_{\bar{\varphi}'\eta\ i(a)} = \bar{\varphi}' i(a). \end{aligned}$$

Therefore, $\varphi = \bar{\varphi}' i$.

$(b) \Rightarrow (a)$. Let $M^{'} = [i(A)]^{*}$ be the free partial monoid on $i(A)$, and let $\eta: \ i(A) \to M^{'}$ be the natural embedding. Let $\varphi: \ i(A) \to M$ be the inclusion map. As in the proof $(a) \Rightarrow (b)$, (since $M^{'}$ is free on $i(A)$), there is a partial monoid homomorphism

$$\Phi : FPM(i(A)) = M^{'} \to M$$

such that $\Phi\eta = \varphi$. That is $\Phi\eta(i(a)) = \varphi(i(a)) = i(a)$, $\quad (a \in A)$, and Φ is the identity on the partial generators of $M^{'}$. Let $\psi : A \to M^{'}$ be given by

$$\psi = \eta i : A \xrightarrow{i} i(A) \xrightarrow{\eta} M^{'}.$$

By (b) there is a partial monoid homomorphism $\Psi : M \to M^{'}$ such that $\Psi i = \psi$. For $i(a) \in M$, we have

$$\Psi(i(a)) = \Psi i(a) = \psi(a) = \eta i(a).$$

Let $x \in M$, $(x = \varepsilon_x)$ be arbitrary, say $a = ia_{j_1} ia_{j_2} ... ia_{j_n} \varepsilon_{\prod_{l=1}^{r} i \ a_l}$ with $a_{j_k} \in \{a_1, ..., a_r\}$, $k = 1, 2, ..., n$ $\quad$ (observe that $i(A)$ partially generates M), and $\varepsilon_x = \varepsilon_{\prod_{l=1}^{r} i \ a_l}$. We have $\Psi x \in M^{'}$ and so $\psi x \in B^{*}$ for some finite set $B = \{b_1, ..., b_r\} \subset i(A)$. Now clearly

$$\varepsilon_{\Psi x} = \varepsilon_{\prod_{l=1}^{r} i \ a_l}$$

which gives

$$\varepsilon_{\Phi\Psi x} = \varepsilon_{\prod_{l=1}^{r} \Phi \ \eta \ i \ a_l} = \varepsilon_{\prod_{l=1}^{r} i \ a_l} = \varepsilon_x.$$

Therefore,

$$\begin{aligned}
\Phi\Psi x &= \Phi(\Psi i \ a_{j_1} ... \Psi i \ a_{j_n}) \Psi \ \varepsilon_x \\
&= \Phi(\eta \ i \ a_{j_1} ... \eta \ i \ a_{j_n}) \varepsilon_{\Psi x} \\
&= \Phi\eta \ i \ a_{j_1} ... \Phi\eta \ i \ a_{j_n} \varepsilon_{\Phi\Psi x} \\
&= \varphi(i \ a_{j_1}) ... \varphi(i \ a_{j_n}) \varepsilon_{\Phi\Psi x} \\
&= i \ a_{j_1} ... i \ a_{j_n} \varepsilon_x = x.
\end{aligned}$$

Thus $\Phi\Psi = id_M$ (the identity map $M \to M$). Likewise, for $x \in M^{'}$, say $x = \eta \ i \ a_{j_1} ... \eta \ i \ a_{j_n} \varepsilon_x$ where ε_x is the identity of the maximal monoid, say B^{*} in $M^{'}$, for some finite subset $B = \{i \ a_1, ..., i \ a_r\}$ of $i(A)$. Thus $\varepsilon_x = \varepsilon_B = \varepsilon_{\eta \ i \ a_1} ... \varepsilon_{\eta \ i \ a_r} = \varepsilon_{\eta \ i \ a_1 ... \eta \ i \ a_r} = \varepsilon_{i \ a_1 ... \ i \ a_r}$. We have

$$\begin{aligned}
\Psi\Phi(x) &= \Psi(\Phi\eta \ i \ a_{j_1} ... \Phi\eta \ i \ a_{j_n} \Phi \ \varepsilon_x) \\
&= \Psi\left(i \ a_{j_1} ... i \ a_{j_n} \ \varepsilon_{\Phi(x)}\right) \\
&= \Psi i \ a_{j_1} ... \Psi i \ a_{j_n} \ \varepsilon_{\Psi\Phi x} \\
&= \eta \ i \ a_{j_1} ... \eta \ i \ a_{j_n} \varepsilon_x = x.
\end{aligned}$$

Thus $\Psi\Phi = id_{M'}$. It follows that M is isomorphic to $M^{'}$ and the proof is complete. $\square$

Given a partial monoid M, we set $S = M - \varepsilon(M)$ and

$$A = \left\{x \in S - S^2 : \varepsilon_x \text{ is maximal in } \varepsilon(M)\right\}$$

Then we have:

Theorem 3.3. *Let M be a partial monoid and let A be the subset of M defined as above. The following two statements are equivalent:*

(a) *M is free on A.*

(b) *For each $x \in M$ with $x \neq \varepsilon_x$, there exists a unique finite set $\{a_1, ..., a_r\} \subset A$ such that $\varepsilon_x = \prod_{i=1}^{r} \varepsilon_{a_i}$ and x has a unique factorization*

$$x = x_1 ... x_m \ \varepsilon_x$$

with $\{x_1, ..., x_m\} \subset \{a_1, ..., a_r\}$.

Proof. $(a) \Rightarrow (b)$. Follows from the definition of a free partial monoid and the property of A.

$(b) \Rightarrow (a)$. Let M' be the free partial monoid on A, and let $\eta : A \to M'$ be the natural embedding. Let

$$\varphi' : A \to M$$

be the inclusion map, that is $\varphi'(a) = a\varepsilon_a = a \quad (a \in A)$. By the universal properly (cf. Theorem 2.1), there exists a unique partial monoid homomorphism

$$\bar{\varphi}' : M' \to M$$

such that $\varphi' = \bar{\varphi}' \circ \eta$. We have $\varphi' : A \to M$ is the inclusion onto the partial generators of M. Define

$$\psi : M \to M'$$

by

$$\psi(x_1 x_2 ... x_m \varepsilon_x) = \eta x_1 \eta x_2 ... \eta x_m \prod_{i=1}^{r} \varepsilon_{\eta x_i}.$$

Clearly ψ is partial monoid homomorphism and $\eta = \psi\varphi'$. Let $b \in M$. By (b), there exists a unique set $\{a_1, ..., a_r\} \subset A$ such that $\varepsilon_b = \prod_{i=1}^{r} \varepsilon_{a_i} = \varepsilon_{\prod_{i=1}^{r} a_i}$ and b has a unique factorization

$$b = b_1 b_2 ... b_n \ \varepsilon_b = b_1 b_2 ... b_n \ \varepsilon_{\prod_{i=1}^{r} a_i}$$

with $\{b_1, b_2, ..., b_n\} \subset \{a_1, ..., a_r\}$. As in the proof $(b) \Rightarrow (a)$ of theorem 2.2 we can show $\varepsilon_{\bar{\varphi}' \psi b} = \varepsilon_b$. Thus we have

$$\begin{aligned}
\bar{\varphi}' \psi b &= \bar{\varphi}' \left(\psi\varphi' b_1 ... \psi\varphi' b_n \ \varepsilon_{\psi b}\right) \\
&= \bar{\varphi}' \ (\eta b_1 ... \eta b_n \ \varepsilon_{\psi b}) \\
&= \bar{\varphi}' \eta b_1 ... \bar{\varphi}' \eta b_n \ \varepsilon_{\bar{\varphi}' \psi b} \\
&= \varphi' b_1 ... \varphi' b_n \ \varepsilon_b = b_1 b_2 ... b_n \ \varepsilon_b = b.
\end{aligned}$$

Thus $\bar{\varphi}' \ \psi = id_M$. Let $b \in M'$, say $b \in B^*$ for some finite $B \subset A$, there exist $b_1, b_2, ..., b_n \in A$ with

$$b = \eta b_1 ... \eta b_n \ \varepsilon_b.$$

We have

$$\begin{aligned}
\psi\bar{\varphi}' \ b &= \psi \ \left(\bar{\varphi}' \eta b_1 \ ... \bar{\varphi}' \eta b_n \ \varepsilon_{\bar{\varphi}' \ b}\right) \\
&= \psi \ \left(\varphi' b_1 ... \varphi' b_n \ \varepsilon_{\bar{\varphi}' \ b}\right) \\
&= \psi\varphi' b_1 ... \psi\varphi' b_n \ \ \varepsilon_{\psi \ \bar{\varphi}' \ b} \\
&= \eta b_1 ... \eta b_n \ \varepsilon_{\psi \ \bar{\varphi}' \ b} \\
&= \eta b_1 ... \eta b_n \ \varepsilon_b = b.
\end{aligned}$$

Thus $\psi\bar{\varphi}' = id_{M'}$. Therefore $M = M'$, and so, M is free on A. ☐

We may conclude directly the following

Corollary 3.4. *Let M be a partial monoid satisfying one of the two equivalent conditions of Theorem 2.3*

(i) *For every $x \in M$, M_{ε_x} is free monoid with finite set of (free) generators $\{a_1 \ \varepsilon_x, ..., a_r \ \varepsilon_x\}$ where $\{a_1, ..., a_r\} \subset A$ is the unique set such that $\varepsilon_x = \prod_{i=1}^{r} \varepsilon_{a_i}$. In particular, for every $x \in A$, M_{ε_x} is cyclic with one generator a, where a is the unique element in A such that $\varepsilon_x = \varepsilon_a$*

(ii) *Every $\varepsilon_x \in \varepsilon(A) = \varepsilon(M)$ has a unique factorization $\varepsilon_x = \prod_{i=1}^{r} \varepsilon_{a_i}$, $a_i \in A$. In particular if $\varepsilon_x = \prod_{i=1}^{r} \varepsilon_{a_i}$ and $\varepsilon_y = \prod_{j=1}^{s} \varepsilon_{b_j}$, a_i, $b_j \in A$ then*

$$\varepsilon_x = \varepsilon_y \quad \text{iff} \quad \{a_i : i = 1, ..., r\} = \{b_j : j = 1, ..., s\}$$

(iii) *For every $\varepsilon_a \geq \varepsilon_b$,*

$$\varphi_{\varepsilon_a, \ \varepsilon_b} : M_{\varepsilon_a} \to M_{\varepsilon_b}$$

is a homomorphism of monoids.

(iv) *For every* $b \in S - S^2$, *there exists a unique* $a \in A$ *such that* $a\ \varepsilon_b = b$.

(v) *For every* $b \in S - S^2$, *we have* $b\ \varepsilon_x \in S - S^2$, *for every* $\varepsilon_x \in \varepsilon(A)$.

References

[1] A. M. Abd-Allah and M. E.-G. M. Abdallah, On Clifford semigroups, Pure Math. Manuscript 7 (1988) 1–17.

[2] A. M. Abd-Allah and M. E.-G. M. Abdallah, Congruences on Clifford semigroups, Pure Math. Manuscript 7 (1988) 19–35.

[3] A. M. Abd-Allah and M. E.-G. M. Abdallah, On tensor products of Cliffored semigroups, Pure Math. Manuscript 7 (1988) 37–52.

[4] A. M. Abd-Allah and M. E.-G. M. Abdallah, Categories of generalized rings associated with partial mappings, Pure Math. Manuscript 7 (1988) 53–65.

[5] A. M. Abd-Allah and M.El-Ghali M. Abdallah, Semilattices of monoids, Indian J. Math. 33 (1991) 325–333.

[6] M. E.-G. M. Abdallah, Algebraic and Topological Properties of Partial Groups, Ph. D. Thesis, Egypt (1988).

[7] M. El-Ghali M. Abdallah, L. N. Gab-Alla and Sayed K. M. Elagan, On semilattices of groups whose arrows are epimorphisms, Int. J. Math. Math. Sci. Volume 2006, Article ID 30673, Pages 1–26.

[8] M. El-Ghali M. Abdallah, L. N. Gab-Alla and Sayed K. M. Elagan, Minimal conditions on Clifford semigroup congruences, Int. J. Math. Math. Sci. Volume 2006, Article ID 76951, Pages 1–9.

[9] A. Brosalina and B. Melnikov, Commutation in Global Supermonoid of Free Monoids, Informatica 11 (2000) 353–370.

[10] S. Y. Chen and Hsieh, Factorizable inverse semigroups, Semigroup Forum 8 (1974) 283–297.

[11] D. Easdown, J. East and D. G. Fitzgerald, Braids and Factorizable inverse monoids, Semigroups and Languages, I.M.Araujo, M.J.J.Branco, V.H.Fernandes and G.M.S.Gomes, World Scientific, (2004) 86–105.

[12] P. M. Higgins, Techniques of semigroup theory, Oxford University Press, Oxford, 1992.

[13] J. M. Howie, An introduction to semigroup theory, Academic Press, London, 1976.

[14] J. M. Howie and N. Ruskuc, Constructions and presentations for monoids, Commun. Algebra 22(15) (1994) 6209–6224.

[15] T. W. Hungerford, Algebra, Springer-Verlag, 1980.

[16] J. Leech, Inverse monoids with a natural semilattice ordering, Proc. Lond. Math. Soc. 70 (1995) 146–182.

[17] G. Lallement, Semigroups and combinatorial applications, Wiley, New York, 1979.

[18] S. MacLane, Categories for the working mathematician, Springer-Verlag, 1971.

[19] M. Petrich, Inverse semigroups, Wiley, 1984.

[20] D. J. S. Robinson, A course in the theory of groups, Springer-Verlag, 1982.

S. K. Elagan (`sayed_khalil2000@yahoo.com`) – Department of Mathematics and Statistics, Faculty of Science, Taif University, Taif, El-Haweiah, P.O. Box 888, Zip Code 21974, Kingdom of Saudi Arabia (KSA)